CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[15] Evolution equations in differential modules.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

1. Definition of differential modules
2. Definition of evolution equations
3. Computations with differential modules
4. Some concluding remarks

Lemma 1.7 in [11] revisited./1

Definition (Differential module)

Let $(\mathbf{k}, \partial)$ a differential ring. A differential module M over $(\mathbf{k}, \partial)$ is a (in general left-) module over $\mathbf{k}\langle\partial\rangle$[a] (see [3] Ch II §1.1). This is equivalent to the data of

1. A $\mathbf{k}$-module M
2. $\partial_M \in End_{\mathbb{Z}}(M)$ such that for all $(a, m) \in \mathbf{k} \times M$
$$\partial_M(a.m) = \partial(a).m + a.\partial_M(m)$$

[a] We note here $\mathbf{k}\langle\partial\rangle$ instead of $\mathbf{k}[\partial]$ as we will have to consider, for instance, $\mathbb{C}[z]\langle\partial\rangle$ for which the notation $\mathbb{C}[z][\partial]$ could be confusing.

Definition (Evolution equation)

Let M be a (finite or infnite dimensional) differential module. We will call evolution equation, in M an expression

$$Y' = \Phi(Y) \text{ with } \Phi \in End_{\mathbf{k}}(M) \tag{1}$$

Lemma 1.7 (revisited)/2

Lemma 1.7 (rev)

Let $M \in \mathbf{Diff_k}$ ($\mathbf{k}$ is a differential field with field of constants $C = \ker(\partial)$) and $\Phi \in End_{\mathbf{k}}(M)$, we suppose that

- $(Y_i)_{i\in I} \in M^I$ is a family of solutions of the some evolution equation of type (1)

Then, TFAE

1. $(Y_i)_{i\in I}$ is C-free
2. $(Y_i)_{i\in I}$ is $\mathbf{k}$-free (for the structure of $\mathbf{k}$-field).

Proof

2 $\Longrightarrow$ 1) being obvious, remains to prove ($1 \Longrightarrow 2$). To this end, let $\mathcal{R}$ be the module of $\mathbf{k}$-linear relations, i.e. we consider the map $\Lambda : \mathbf{k}^{(I)} \to M$ defined by $(Y_i)_{i\in I}$ [a] such that $\Lambda(\alpha) = \sum_{i\in I} \alpha(i) Y_i$ (then $\mathcal{R} = \ker(\Lambda)$).

[a] See [3] ch II §1.11 def 10.

Proof of (revisited Lemma) 1.7 cont'd

Either $\mathcal{R} = \{0\}$ and we are done or $\mathcal{R} \neq \{0\}$. In this case, we take $\beta \in \mathcal{R} \setminus \{0\}$ with minimal[a] support $F \neq \emptyset$ and $i_0 \in F$.
Due to the fact that **k** is a field, we can take $\beta(i_0) = 1$. Then

$$(LR) \qquad Y_{i_0} + \sum_{i \in F \setminus \{i_0\}} \beta(i) Y_i = 0$$

$$(\partial) \qquad Y'_{i_0} + \sum_{i \in F \setminus \{i_0\}} \beta(i) Y'_i + \beta(i)' Y_i = 0$$

$$(\Phi) \qquad \Phi(Y_{i_0}) + \sum_{i \in F \setminus \{i_0\}} \beta(i) \Phi(Y_i) = 0$$

[a] For cardinality or inclusion.

Lemma 1.7/3

We perform (2)-(3) (repeated below)

$$(\partial) \qquad Y'_{i_0} + \sum_{i \in F \setminus \{i_0\}} \beta(i) Y'_i + \beta(i)' Y_i = 0 \tag{2}$$

$$(\Phi) \qquad \Phi(Y_{i_0}) + \sum_{i \in F \setminus \{i_0\}} \beta(i) \Phi(Y_i) = 0 \tag{3}$$

and get $\sum_{i \in F \setminus \{i_0\}} \beta(i)' Y_i = 0$. But, as F is minimal, the family $(Y_i)_{i \in F \setminus \{i_0\}}$ is **k**-free. This entails $\beta(i)' = 0$ for all $i \in F \setminus \{i_0\}$ and then $\beta(i) \in C$, from hypothesis, we get $F \setminus \{i_0\} = \emptyset$ and $F = \{i_0\}$ i.e. $Y_{i_0} = 0$. This is impossible because of hypothesis 1 $((Y_i)_{i \in I}$ is C-free). CQFD

Example 1: Vector fields on the line.

1. (Vector fields on the line) In physics and computer science literature, there is a lot of confusion between evolution equations,
   a) **well defined ?**
   b) **that can be stated ?**
   c) **that can be integrated ?**

2. These problems can be cured
   a,b) making precise the spaces and transformations $\mathbf{\Phi}$.
   c) examining the conditions of integration.

3. A banal and trite commonplace is the formula

$$e^{t\frac{d}{dx}}(f)[x] = f[x+t] \tag{4}$$

freely and lightheartedly repeated everywhere which soon becomes as a philosophical moto *"Exponential of $t\times$ derivation"="displacement by t"*.

Example 1: Vector fields on the line./2

4. This formula is true in some frameworks and false in others. Let $D \in \mathfrak{der}(\mathcal{A})$ where $\mathcal{A}$ is some (associative) algebra. The evolution equation reads

$$\frac{d}{dt}(Y) = Y' = t.D.Y \ ; \ Y \in \mathcal{A} \subset End(\text{some space}) \tag{5}$$

5. Firstly, if D is locally nilpotent $\exp(t.D)$ is a one-parameter group of automorphisms of $\mathcal{A}$ (Ex. $\mathcal{A} = \mathbb{C}[x]$, $D = \frac{d}{dx}$ leads to formula 4).
6. With $\mathcal{A} = C^\infty(\mathbb{R}, \mathbb{R})$ the evolution equation **can be stated** i.e. the two members of (4) are well-defined, but this formula is false (take any Schwartz test function).
7. With $\mathcal{A} = C^\omega(\mathbb{R}, \mathbb{R})$, the formula is true.

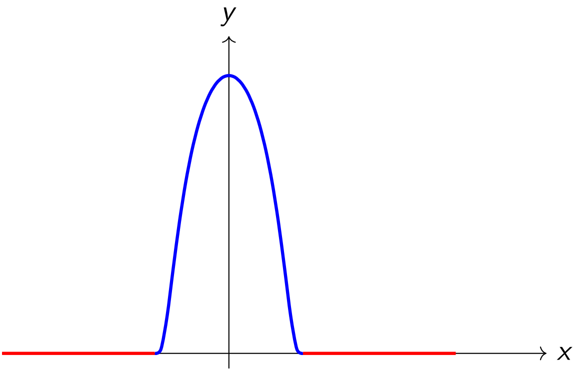


Figure: Schwartz C^{ω} test function. Piecewise defined: for $x \notin]-1,-1[$, $f(x) = 0$ (red) and for $x \notin]-1,-1[$, $f(x) = 10\exp(\frac{1}{x^2-1})$ (blue). Formula (4) is false. Indeed for every x in the red domain and $t = -x$, in

$$e^{t\frac{d}{dx}}(f)[x] = f[x+t].$$

we have $LHS(t,x) = 0$ whereas $RHS(x,t) = 10$.

Example 1: Vector fields on the line./3

1. For suitable spaces (another talk, see also [6])

$$e^{t.x\frac{d}{dx}}(f)[x] = f[e^t.x] \qquad e^{t.x^2\frac{d}{dx}}(f)[x] = f[\frac{x}{1-t.x}]$$

$$e^{t.x^3\frac{d}{dx}}(f)[x] = f[\frac{x}{\sqrt{1-2.t.x^2}}] \qquad e^{t.x^{r+1}\frac{d}{dx}}(f)[x] = f[\frac{x}{\sqrt[r]{1-t.r.x^r}}]$$

Application (Van der Put).

4. Let (R, ∂) be a (commutative) differential ring, containing the differential field $\mathbf{k}$ (we suppose[a] $C = \ker(\partial) \subset \mathbf{k}$). We consider $L \in \mathbf{k}\langle\partial\rangle$ of degree n.

$$L = a_n.\partial^n + \ldots + a_1.\partial + a_0 \tag{6}$$

then, if R is without zero divisors, the set of solutions[b] $L.y = 0$ is a $\mathbf{k}$-vector space of dimension $\leq n$.

[a] This is not granted (Ex. $R = \mathbb{C}[x, y], \partial = \frac{d}{dx}, \mathbf{k} = \mathbb{C}, C = \mathbb{C}[y]$).
[b] $Soln_R(L)$ in [11].

Proof. – (Sketch) Embed $R \hookrightarrow Frac(R)$ as differential rings and apply [11] Lemma 1.10.

Remark. – The result is no longer true if R has zero divisors (see below).
Example. – $R = (\mathcal{H}(\Omega), \partial)$ where Ω is not connected. Take $\Omega = \Omega_1 \cup \Omega_2$ (Ω_i connected components) and $y'' + y = 0$, the space $Soln_R(L)$ is of dimension 4. With basis $[cos(z).1_{\Omega_1}, sin(z).1_{\Omega_1}, cos(z).1_{\Omega_2}, sin(z).1_{\Omega_2}]$.

Counterexamples cont'd

5. One could argue that, in the preceding example, the ring of constants is not a field. Indeed
$$\ker(\partial) = \mathbb{C}.1_{\Omega_1} \oplus \mathbb{C}.1_{\Omega_2}$$

6. We now consider an example coming from algebraic geometry (coordinate ring of $xy = 0$). Let us consider, in $\mathbb{C}[x, y]$, the ideal $\mathcal{J}_{xy}$ generated by xy (it has $\{x^p y^q\}_{p,q\geq 1}$ as a basis) and $\partial = x\frac{d}{dx} + y\frac{d}{dy}$. One can check that $\mathcal{J}_{xy}$ is a differential ideal for ∂. Then
$$\mathcal{A} = \mathbb{C}[x, y] \Big/ \mathcal{J}_{xy} = \mathbb{C}.1 \oplus \mathbb{C}_+[x] \oplus \mathbb{C}_+[y]$$
is a differential algebra. For $N \geq 1$, the equation
$$0 = Y' - N.Y = (\partial - N).y$$
has a two dimensional $\mathbb{C}$-vector space of solutions
$$V = \mathbb{C}.x^N \oplus \mathbb{C}.y^N$$

Examples cont'd and unfold

7. In order to obtain correct arrows for the set annihilated by $\partial - 1$ we would have to localize by the wronskian of one set of solutions of $dim = 1$ (here $\{x\}$ or $\{y\}$) but each of these wronskians are zero divisors, so the correct theory will imply
    1. Non-zero divisors
    2. Normalization (monic differential operator and localization by wronskian). We go back to our favorite module $M(x_0^2 x_1 x_0 x_1)$ generated by the full vector space of solutions of $L_{x_1 x_0 x_1 x_0^2}.y = 0$ of dimension 6.
8. Now, we will need a lemma which is very useful to compute complicated wronskians.

An exponential-like trace phenomenon.

Definition (Fundamental matrix) to be revised

1. In the context of slide 5, let us suppose that M is finite-dimensional. We will say that a finite family $(Y_i)_{i\in I}$ is **fundamental** if it is a $\mathbf{k}$-basis of $Soln_{\mathbf{k}}(L) \cap M$.
2. In the case when $M = \mathbf{k}^n$, $I = \{1, \cdots, n\}$ and $Y_i =^t (y_i^1, \cdots, y_i^n)$, the matrix whose colunms are the Y_i i.e.

$$\begin{pmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^n & \cdots & y_n^n \end{pmatrix}$$

will be called a fundamental matrix for $Y' = A.Y$ (where A is the matrix of $F \in End_{\mathbf{k}}(M)$ in the canonical basis).

Examples cont'd and unfold/2

9. The wronskian

$$W = wr(1, \mathrm{Li}_{x_1}, \mathrm{Li}_{x_0x_1}, \mathrm{Li}_{x_1x_0x_1}, \mathrm{Li}_{x_0x_1x_0x_1}, \mathrm{Li}_{x_0^2x_1x_0x_1})$$

is the determinant of the matrix

$$M_{x_0^2x_1x_0x_1} = \begin{bmatrix} 1 & \mathrm{Li}_{x_1} & \mathrm{Li}_{x_0x_1} & \mathrm{Li}_{x_1x_0x_1} & \mathrm{Li}_{x_0x_1x_0x_1} & \mathrm{Li}_{x_0^2x_1x_0x_1} \\ 0 & (1-z)^{-1} & *\,\mathrm{Li}_{x_1} & *\,\mathrm{Li}_{x_0x_1} & *\,\mathrm{Li}_{x_1x_0x_1} & *\,\mathrm{Li}_{x_0x_1x_0x_1} \\ 0 & (1-z)^{-2} & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (7)$$

10. This matrix is fundamental for the $\mathbb{C}(z)^6$ evolution equation. $Y' = A_{D_{x_1x_0x_1x_0^2}} Y$ and it satisfies $W' = tr(A_{D_{x_1x_0x_1x_0^2}})W$ please check ! (see, [11] Ex. 1.14.5, with $D_w = z^{-|w|_{x_0}}(1-z)^{-|w|_{x_1}} L_w$).
$W' = \dfrac{15z-11}{z(1-z)} W = \dfrac{11(z-1)+4z}{z(1-z)} W = (\dfrac{-11}{z} + \dfrac{4}{1-z})W$ whence

$$W = \lambda_1.exp(\int \frac{4}{1-z} - \frac{11}{z}) = \lambda_2.\frac{1}{(1-z)^4}.\frac{1}{z^{11}}$$

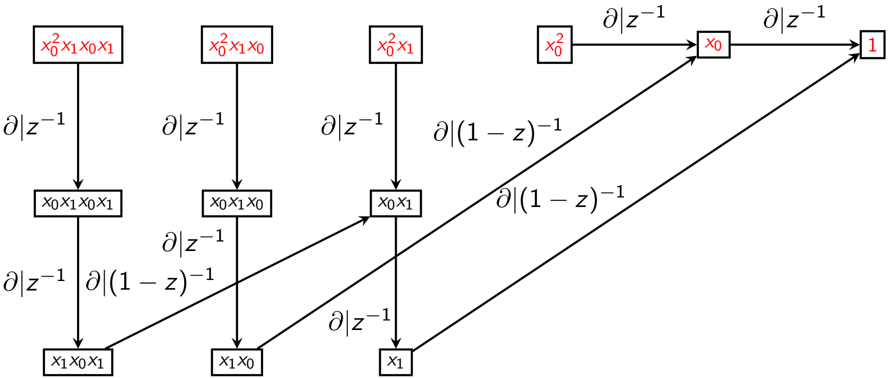


Figure: Structure of the differential module M_w for $w = x_0^2x_1x_0x_1$. The $\mathbb{C}$-vector space V_w of all solutions of $L_{\widetilde{w}}.y = 0$ is in red and actions of ∂ are marked edges with multiplicities after `mid` i.e. `<action>|<multiplicity>`

The nodes form a $\mathbb{C}(z)$-basis of the module i.e. the universal module of all solutions of $L.y = 0$ with

$L = \partial\theta_{x_1x_0x_1x_0^2} = L_{x_1x_0x_1x_0^2} = (z^5 - 2z^4 + z^3)\partial^6 + (15z^4 - 26z^3 + 11z^2)\partial^5 + (65z^3 - 93z^2 + 30z)\partial^4 + (90z^2 - 97z + 18)\partial^3 + (31z - 20)\partial^2 + \partial$

A cyclic module

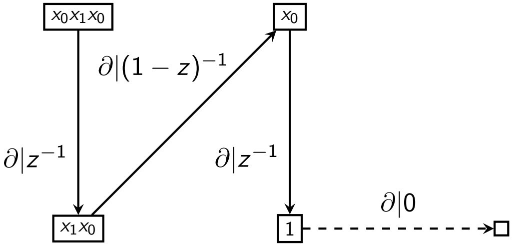


Figure: Cyclic (or monogeneous) differential module $M = \mathbb{C}(z)\langle\partial\rangle\,\mathrm{Li}_w$ for $w = x_0x_1x_0$. Note that it does not contain the $\mathbb{C}$-vector space V_w of all solutions of $D_w = \partial\theta_{\tilde{w}}.y = 0$. Actions of ∂ are marked edges multiplicities are after `mid` i.e. `<action>|<multiplicity>` one has $L = \partial\theta_{x_0x_1x_0} =$ $(z^4 - 2z^3 + z^2)\partial^5 + (10z^3 - 16z^2 + 6z)\partial^4 + (25z^2 - 29z + 6)\partial^3 + (15z - 10)\partial^2 + \partial$

A model evolution equation (generalized BTT).

1. Let $(\mathcal{A}, \partial)$ be a differential algebra over $\mathbf{k} = \ker(\partial)$ and a differential field $\mathcal{C} \supset \mathbf{k}$. We consider an alphabet X and a family of $(u_x)_{x \in X}$ of "inputs". We form the "multiplier" $M = \sum_{x \in x} u_x\, x \in \widehat{\mathbb{C}.X}$ and consider the evolution equation in $\mathcal{A}\langle\langle X \rangle\rangle$

$$\mathbf{d}(S) = M.S \; ; \; \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \tag{8}$$

where $\mathbf{d}$ is the termwise differentiation

$$\mathbf{d}(S) := \sum_{w \in X^*} \partial(\langle S | w \rangle)\, w \tag{9}$$

A model evolution equation (generalized BTT).

2. Under the preceding conditions 1 , we have the following

BTT theorem

The following are equivalent

i. The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.
ii. The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.
iii. The family $(u_x)_{x\in X}$ is such that, for $f\in\mathcal{C}$ and $\alpha\in k^{(X)}$ (i.e. $supp(\alpha)$ is finite)

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{10}$$

iv. The family $(u_x)_{x\in X}$ is free over k and

$$d(\mathcal{C})\cap \mathrm{span}_k\left((u_x)_{x\in X}\right)=\{0\}\ . \tag{11}$$

Picard's process

3. Note that such solutions can be considered as paths $S(z)$ drawn on the Magnus group (this is more apparent with $\mathcal{A}=\mathcal{H}(\Omega)$ or $\mathcal{A}=C^\infty(\Omega;\mathbb{R})$). Conversely, every $\mathcal{A}$-path drawn on the Magnus group are solutions of some system

$$\mathbf{d}(S)=M.S\ ;\ \langle S|1_{X^*}\rangle=1_{\mathcal{A}} \tag{12}$$

with $M\in\mathcal{C}_+\langle\langle X\rangle\rangle$ for some $\mathcal{C}$ a subalgebra of $\mathcal{A}$ (in fact $\mathcal{C}$ includes the smallest subalgebra containing the coefficients of $\mathbf{d}(S)S^{-1}$).

4. Conversely, if $M\in\mathcal{C}_+\langle\langle X\rangle\rangle$ for some $\mathcal{C}$ a subalgebra of the differential algebra $(\mathcal{A},\partial)$ with a section[a] $\int$, one can construct a solution of (12) by the Picard's process. One computes the limit $\lim_{n\to+\infty}S_n$ where

$$S_0=1_{X^*}\ ;\ S_{n+1}=1_{X^*}+\int M.S_n\ . \tag{13}$$

[a] Not all differential algebras possess such a section (as $\mathbb{C}(z)$ for instance).

About sections

5. The best example of section is $\int_{z_0}^{z}$. Let Ω be a domain (i.e. connected open subset) of $\mathbb{C}$
6. Then $f \mapsto \int_{z_0}^{z} f(s)ds$ is a section for $(\mathcal{H}(\Omega), \frac{d}{dz})$
7. For this section, Picard's process applied to the NcEvEq (noncommutative evolution equation)
$$\mathbf{d}(S) = M.S;\ \langle S|1_{X^*}\rangle = 1_{\mathcal{A}};\ M = \sum_{x\in X} u_x\, x$$
has for limit $S = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w$.
8. It is sometimes hepful to use other (more adapted) integrators which should always (I insist) be considered with care, i.e. with their domains.

Reserve : Application to the freeness of solutions of $L.y = 0$

In fact, let us consider $y_1, \ldots, y_n$ a set of $C = \ker(\partial)$-independent solutions of $L.y = 0$. Stringing ${}^t(y, y', \ldots, y^{n-1})$, we get a set of solutions of

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{n-1} \end{pmatrix}' = CompMat(L) \begin{pmatrix} y \\ y' \\ \vdots \\ y^{n-1} \end{pmatrix} \tag{14}$$

where $CompMat(L)$ is the companion matrix of L, we can apply the lemma to see that the strings ${}^t(y_i, y_i', \ldots, y_i^{n-1})$ are all linearly independent over $\mathbf{k}$ and then so are $y_1, \ldots, y_n$. This can be, in particular, applied to the modules of polylogs.

A simple transition system: weighted graphs

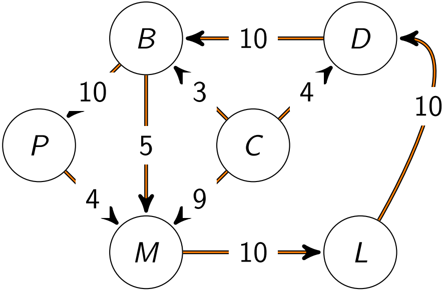


Figure: Directed graph weighted by numbers which can be lengths, time (durations), costs, fuel consumption, probabilities. This graph is equivalent to a square matrix. Coefficients are taken in different semirings (i.e. rings without the "minus" operation, as tropical or [max,+]) according to the type of computations to be done. Tropical mathematics were so called by MPS school because they were founded by the Hungarian-born Brazilian mathematician and computer scientist Imre Simon.

A small tribute to MPS or Marco as we used to call him

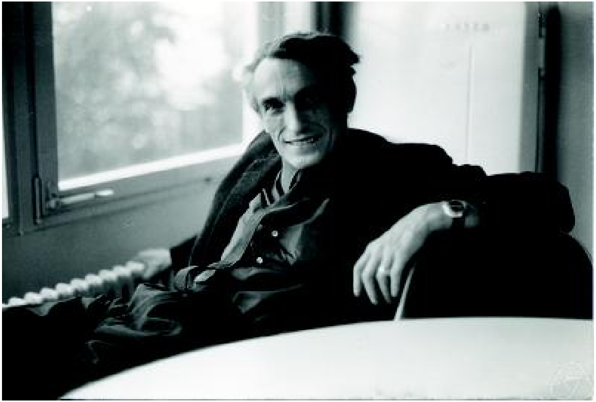
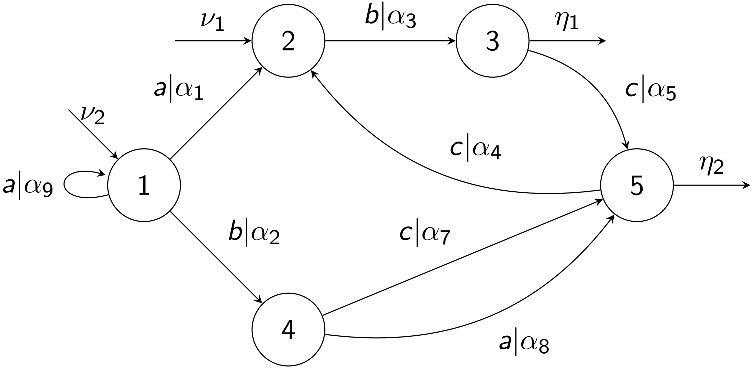


Figure: Marcel-Paul Schützenberger at Oberwolfach (1973)[1]

[1] Contrary to 1972 (Wikipedia)

Multiplicity Automaton (Eilenberg, Schützenberger)



1 S. Eilenberg, *Automata, Languages, and Machines (Vol. A)* Acad. Press, New York, 1974

2 M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.

Multiplicity automaton (linear representation) & behaviour

Linear representation

$$\nu = \begin{pmatrix} \nu_2 & \nu_1 & 0 & 0 & 0 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 0 & \eta_1 & 0 & \eta_2 \end{pmatrix}^T$$

$$\mu(a) = \begin{pmatrix} \alpha_9 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

Behaviour

$$\mathcal{A}(w) = \nu\,\mu(w)\,\eta = \sum_{\substack{i,j \\ \text{states}}} \nu(i) \underbrace{\left(\sum weight(p)\right)}_{\substack{\text{weight of all paths } (i) \to (j) \\ \text{with label } w}} \eta(j)$$

Construction starting from a series S (and actions x^{-1}).

- **States** $\boxed{u^{-1}S}$ (constructed step by step)
- **Edges** We shift every state by letters (length) level by level (knowing that $x^{-1}(u^{-1}S) = (ux)^{-1}S$). Two cases:
  Returning state: The state is a linear combination of the already created ones i.e. $x^{-1}(u^{-1}S) = \sum_{v\in F} \alpha(ux, v)v^{-1}S$ (with F finite), then we set the edges

  $$\boxed{u^{-1}S} \xrightarrow{x|\alpha_v} \boxed{v^{-1}S}$$

  The created state is new: Then

  $$\boxed{u^{-1}S} \xrightarrow{x|1} \boxed{x^{-1}(u^{-1}S)}$$

- **Input** $\boxed{S}$ with the weight 1
- **Outputs** All states $\boxed{T}$ with weight $\langle T|1_{X^*}\rangle$

From theory to practice: Schützenberger's calculus

From series to automata

Starting from a series S, one has a way to construct an automaton (finite-stated iff the series is rational) providing that we know how to compute on shifts and one-letter-shifts will be sufficient due to the formula $u^{-1}v^{-1}S = (vu)^{-1}S$.

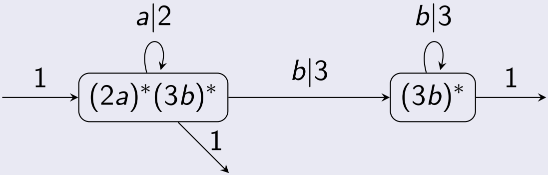
Calculus on rational expressions ([1], lemma 7.2).

In the following, x is a letter, E, F are rational expressions (i.e. expressions built from letters by scalings, concatenations and stars)

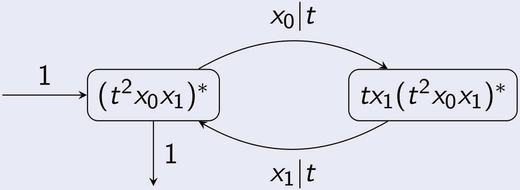
1. x^{-1} is (left and right) linear
2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E|1_{X^*}\rangle x^{-1}(F)$
3. $x^{-1}(E^*) = x^{-1}(E).E^*$

Computations with “returning states”.

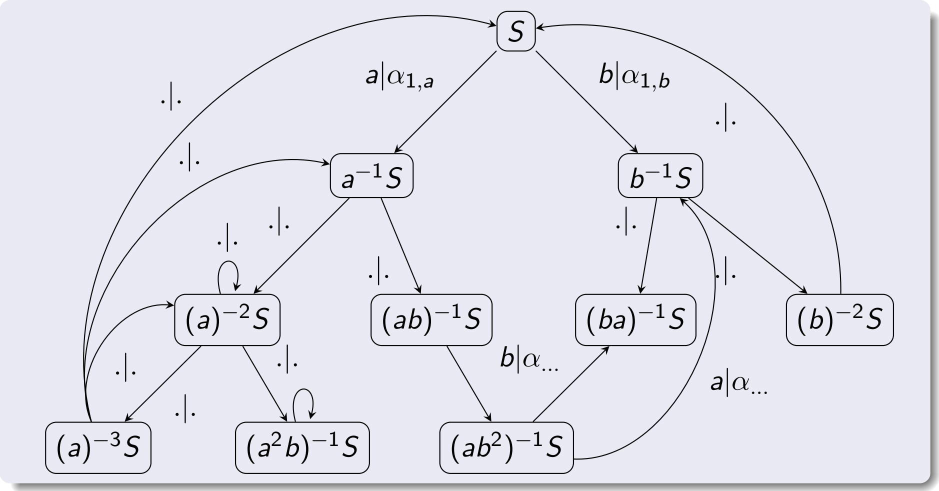
With $(2a)^*(3b)^*$; $X = \{a, b\}$



With $(t^2x_0x_1)^*$; $X = \{x_0, x_1\}$



In general: returning edges



Use of this transition structure

Automata with multiplicities is an elegant way to code

- Algebraic numbers
- Continued fractions (quadratic irrationalities, Lagrange's theorem, see Knuth)
- Markov chains (several transition matrices)
- Finite-length (e.g. finite-dimensional one the ground field of the algebra) modules
- In particular differential modules

Constructions with differential modules

Let $(\mathbf{k}, \partial)$ be a differential algebra (over its ring of constants $C = \ker(\partial)$). Recall that

1. The C-algebra of differential operators is
$$\mathbf{k}[\partial] = \mathbf{k} *_{\mathbb{Z}} \mathbb{Z}[\partial] \Big/ (\partial.a - (a' + a.\partial))$$
2. (Normal form) Every element L of $\mathbf{k}[\partial]$ expresses uniquely as
$$L = \sum_{j=0}^{n} a_j \partial^j \text{ with } a_j \in \mathbf{k}$$
3. Note that $\mathbf{k}[\partial]$is only a $\mathbf{k}$-bimodule and NOT a $\mathbf{k}$-algebra (only a C-algebra).
4. There is a euclidean division (but one must precise if it is left or right). Same thing for the extended euclidean algorithm.
5. A differential module M over $(\mathbf{k}, \partial)$ is simply a (in general left-) module over $\mathbf{k}[\partial]$. This is equivalent to the data of
   1. A $\mathbf{k}$-module M
   2. $\partial_M \in End_{\mathbb{Z}}(M)$ such that for all $(a, m) \in \mathbf{k} \times M$
$$\partial_M(a.m) = \partial(a).m + a'.\partial_M(m)$$

Representations

1. **k**-differential modules for a category (see [11] p44 " The category of all differential modules over **k** will be denoted by $\mathbf{Diff_k}$").
2. On a graphical level, a differential f.g. module M can be represented as a marked graph (only the transition structure of an automaton i.e. without initial and final states)
   1. A set of states (elements of a generating set)
   2. Transitions

$$\text{p} \xrightarrow{\partial|\alpha} \text{q}$$

3. Then, we can use the richness of constructions of automata theory to concretely compute with differential modules.
4. Mainly, we can do: direct sums, quotients, tensor products, and (various) shuffle products of automata.
5. Let us first make precise what is a linear representation of an automaton. It is due to the following theorem (Abe, Sweedler).

Rational series (Sweedler & Schützenberger)

Theorem A

Let $S \in \mathbf{k}\langle\langle X\rangle\rangle$ TFAE
i) The family $(Su^{-1})_{u\in X^*}$ is of finite rank.
ii) The family $(u^{-1}S)_{u\in X^*}$ is of finite rank.
iii) The family $(u^{-1}Sv^{-1})_{u,v\in X^*}$ is of finite rank.
iv) It exists $n \in \mathbb{N}$, $\lambda \in k^{1\times n}$, $\mu : X^* \to k^{n\times n}$ (a multiplicative morphism) and $\tau \in k^{n\times 1}$ such that, for all $w \in X^*$

$$(S, w) = \lambda\mu(w)\tau \qquad (15)$$

v) The series S is in the closure of $\widehat{\mathbf{k}.X}$ for $(+, conc, ^*)$ within $k\langle\langle X\rangle\rangle$.

Definition

i) A series S which fulfills one of the conditions of Theorem A will be called *rational*. The set of these series will be denoted by $k^{rat}\langle\langle X\rangle\rangle$.
ii) The triple (λ, μ, τ) as in (15) is called a linear representation of S.

Concluding remarks

1. The category $\mathbf{Diff_k}$ of differential modules has many properties in common with transition structures emerging from automata theory (direct sums, quotients and tensor products which the law of shuffle products).
2. Evolution equations is a wide domain still under development with all kinds of tools (some rigorously, some loosely defined) that we can inherit from combinatorial physics and adapt to our situation.
3. Modules $M(w)$ from polylogarithms give a first example of concrete studies
4. Other modules can be obtained from coordinates of solutions of $S' = M.S$ with $M \in \mathbb{C}_+\langle\langle X\rangle\rangle$ (next time together with generalized BTT).

Concluding remarks/2

5. Indeed, our finite-dimensional differential modules are torsion modules.
   https://en.wikipedia.org/wiki/Torsion_(algebra)
6. For these modules Lam's theorem (2007, not very difficult but very deep categrically speaking) (see [9] Ex. 10.19 p233) is central and connects their category with Ore rings.
7. Next time, more on combinatorics of cyclic modules.

THANK YOU FOR YOUR ATTENTION !

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